PINCHING OF A HIGH-CURRENT RELATIVISTIC CHARGE-NEUTRALIZED ELECTRON BEAM

A. V. Zharinov and A. S. Chikhachev

UDC 533.951

One of the important factors limiting the possibility of obtaining narrow high-current beams is the relatively low current density obtainable from the emitter. When the emitting surface is increased to achieve larger values of the total beam current under conditions ensuring complete charge compensation, pinching by external fields may turn out to be the only means of obtaining narrow high-current electron beams. The theoretical description of a high-current beam is complicated by the necessity of taking account of the self-magnetic field, which may be larger than the external focusing fields. This complexity can be overcome by constructing rather simple models which take account of the interaction of the beam particles with the self-field.

Let us consider a steady axisymmetric relativistic electron beam when the transverse motion of the particles can be separated from the longitudinal, i.e. $\beta_\perp < 1/\gamma_0$, $\gamma_0 = (1 - \beta_0^2)^{-1/2}$. The beam current is assumed much smaller than the Alfvén critical value ($\beta_z \approx \beta_0$). The electron distribution function can be written in the form $F_e = \delta(\beta_z - \beta_0)f(r, z, \mathbf{v}_\perp)$. For f we have the equation

$$c\beta_0 \frac{\partial f}{\partial z} + \mathbf{v}_\perp \frac{\partial f}{\partial \mathbf{r}_\perp} - \left(\omega_e^2 \mathbf{r}_\perp + \frac{2eJ_0\beta_0}{ca^2}\mathbf{r}_\perp + [\mathbf{v}_\perp \boldsymbol{\omega}_H] + \frac{1}{2} r \frac{d\omega_H}{dz}[\mathbf{v}_z \mathbf{r}_\perp]\right)\frac{\partial f}{\partial \mathbf{v}_\perp} = 0, \quad (1)$$

where $\beta_z = v_z/c$, v_z is the axial component of the electron velocity, c is the speed of light, γ is the relativistic factor, $\omega_e^2 = (2\pi e^2/m\gamma)(n_i - n_e/\gamma^2)$, e and m are the charge and mass of the electron, n_i and n_e are the ion and electron densities in the beam (for a neutral beam $n_i = n_e$ and $\omega_e^2 = 2eJ\beta_0/\gamma mcR^2$), J is the beam current, R(z) is the radius of the beam, $\boldsymbol{\omega}_H = e\mathbf{H}/\gamma mc$, where H is the external magnetic field ($\mathbf{H} = \mathbf{e}_z H_z(z)$), J_0 is the external current uniformly distributed over a cross section of radius a and flowing along the axis of the beam (a R(z)).

An uncharged beam can be focused only by magnetic fields, and two configurations must be distinguished: the magnetic field of the external current, which has only an angular component H_φ, and the field of the solenoid $H = H_z(z)$. The density of the beam electrons is assumed constant over the cross section.

It is convenient to rewrite Eq. (1) in the following variables: v_r — the radial component of velocity, M — the moment of the transverse velocity with respect to the beam axis, and r, φ — the distance from the axis and the angle in the $\mathbf{r}_\perp$ plane. The result has the form

$$c\beta_0 \frac{\partial f}{\partial z} + v_r \frac{\partial f}{\partial r} + \frac{M}{r^2}\frac{\partial f}{\partial \varphi} + \frac{M^2}{r^3}\frac{\partial f}{\partial v_r} - \omega_1^2 r \frac{\partial f}{\partial v_r} - \omega_H\left(r v_r \frac{\partial f}{\partial M} - \frac{M}{r}\frac{\partial f}{\partial v_r}\right) - \frac{c\beta_0}{2}\frac{\partial f}{\partial M} r^2 \omega_H' = 0, \quad (2)$$

$$\omega_1^2 = \frac{2eJ\beta_0}{\gamma mcR^2} + \frac{2eJ_0\beta_0}{\gamma mca^2}.$$

Equation (2) is equivalent to the system

$$\frac{dz}{c\beta_0} = \frac{dr}{v_r} = \frac{r^2 d\varphi}{M} = \frac{dv_r}{\frac{M^2}{r^3} - \omega_1^2 r + \frac{M\omega_H}{r}} = -\frac{dM}{\omega_H r v_r + \frac{1}{2} r^2 c\beta_0 \omega_H'},$$

from which we have

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 4, pp. 3-10, July-August, 1980. Original article submitted May 25, 1979.

0021-8944/80/2104-0433$07.50

$$\frac{d^2r}{dz^2} = \frac{1}{(c\beta_0)^2}\left(\frac{M^2}{r^3} - \omega_1^2 r + \frac{M\omega_H}{r}\right), \quad \frac{dM}{dz} = -\omega_H r\frac{dr}{dz} - \frac{1}{2}r^2\frac{d\omega_H}{dz}.$$

From the last equation we can obtain

$$M(z) = M_0 - \omega_H r^2/2,$$

from which we then have for r"

$$(c\beta_0)^2 r'' = \frac{M_0^2}{r^3} - r\omega_2^2(r), \quad \omega_2^2 = \omega_1^2 + \frac{\omega_H^2}{4}. \tag{3}$$

Equation (3) has an invariant of the form

$$I = A(z)\left(\left(r' - \frac{A'}{2A}r\right)^2 + \frac{M_0^2}{c^2\beta_0^2 r^2}\right) + \frac{E_0^2 r^2}{A(z)}, \tag{4}$$

where E_0 is a constant whose meaning will become clear later, and A(z) satisfies the equation

$$\frac{A''}{2A} = -\frac{\omega_2^2(z)}{c^2\beta_0^2} + \frac{A'^2}{4A^2} + \frac{E_0^2}{A^2}. \tag{5}$$

It can be shown by direct substitution that I given by (4) satisfies Eq. (2). Consequently $\Psi(I)$, where Ψ is an arbitrary function, will also satisfy this equation.

We set $\Psi(I) = \varkappa\,\delta(I - I_0)$. Then

$$n_e = \varkappa\int\delta(I - I_0)\,d\mathbf{v}_\perp = \varkappa\int dv_r\frac{dM}{r}\delta(I - I_0) = \frac{\pi\varkappa}{(c\beta_0)^2}\frac{1}{A}\sigma(R - r), \tag{6}$$

$$\sigma(x) = \begin{cases} 0, & x < 0, \\ 1, & x > 0; \end{cases}$$

$$R^2(z) = \frac{I_0}{E_0^2}A(z). \tag{7}$$

It is clear from these relations that R(z) is the radius of the beam.

Substituting the expression for A(z) from (7) into (4), we have

$$1 = \frac{r^2}{R^2} + R^2\left(\frac{E_0}{I_0}\right)^2\left[\left(\frac{v_r}{c\beta_0} - \frac{R'r}{R}\right)^2 + \frac{M_0^2}{c^2\beta_0^2 r^2}\right]. \tag{8}$$

It follows from this equation that if $r = R$, $(v_r/c\beta_0 - R')^2 + M_0^2/c^2\beta_0^2R^2 = 0$; i.e., an electron at the edge of the beam has a moment $M_0 = 0$ and a velocity $v_r = c\beta_0 R'$; in the absence of a magnetic field only those particles which intersect the axis reach the edges of the beam.

The boundary conditions at the beam entrance (z = 0) can also be found from Eq. (8). Setting $R'(0) = 0$, we have

$$v_\perp^2 + \mathrm{const}\, r^2 = \mathrm{const}\, R^2\,(\mathrm{const} > 0),$$

i.e. the perpendicular component of the velocity of the particles must be maximum on the axis of the beam and vanish at its edge. We note that $\mathbf{v}_\perp$ and $c\,\mathbf{r}_\perp$ do not generally have the same direction. The random character of the direction of $\mathbf{v}_\perp$ can be interpreted as the presence in the beam of a transverse "temperature" which varies over the cross section from a maximum on the axis to zero at the edge of the beam. This last remark clearly shows the model character of the description of the beam being considered.

From (5) and (6) we have the equation for the radius

$$(c\beta_0)^2 R'' + \frac{2eJ\beta_0}{\gamma mc}\frac{1}{R} + \frac{2eJ_0\beta_0 R}{\gamma mca^2} + \frac{e^2H^2}{4(\gamma mc)^2}R - \frac{I_0^2}{E_0^2R^3} = 0. \tag{9}$$

Equations of the type (9) were derived earlier in a number of papers [1-3]. Kapchinskii [1] gave the derivation closest to ours by using a microcanonical distribution. Poukey and Toepfer [2] give a hydrodynamic derivation. By averaging over the particles Cooper et al. [3, 4] derived "mean-square" equations similar to (9). In all these papers a coefficient corresponding to I_0^2/E_0^2 in (9) is interpreted as the emittance of the beam, i.e., the phase volume in the transverse coordinates r and r'.

It is clear from (9) that focusing an axisymmetric beam by a longitudinal current is equivalent to focusing by a longitudinal magnetic field, since the third and fourth terms in this equation have the same linear dependence on R. The equivalence relation can be written in the form $H_{eq} = 8J_0\beta_0\gamma mc/ea^2$. We note that the term with the magnetic field enters everywhere with a positive sign ($\sim e^2H^2$), whereas the longitudinal current, directed opposite to the beam current, has a demagnetizing character, which is important, as we shall see later, for a "narrow" pinch.

The problem of focusing, or more accurately, the pinching of a high-current relativistic electron beam was treated, for example, by Poukey et al. [2, 5]. In the case studied in [2] when $i = 2eJ/\gamma mc^3\beta_0 > 1$ it was necessary to use very complicated methods for describing the beam, and to make extensive use of numerical calculations. The case $i < 1$ was treated in [5], but in our opinion certain aspects of the pinching of a neutralized beam with a current smaller than the Alfvén critical value remain obscure.

It is convenient to rewrite Eq. (9) for the radius of the beam in the form

$$R'' + \frac{i}{R} + \frac{i_0' R}{a^2} - \frac{iR_0^2}{R^3} = 0, \tag{10}$$

where

$$i = \frac{2eJ}{\gamma mc^3\beta_0}; \quad i_0' = \frac{2eJ_0}{\gamma mc^3\beta_0}; \quad R_0^2 = \frac{I_0^2}{iE_0^2}; \quad H = 0;$$

R_0 is the equilibrium radius of the beam in the absence of external fields. If $i_0' \equiv 0$, the phase trajectories of the beam in the coordinates R and R' surround the point $R' = 0$, $R = R_0$ (Fig. 1) $(R'^2/2 + i\ln(R/R_0) + (i/2)(R_0^2/R^2) = c_0^2)$.

The character of the quantity R_0 can also be illustrated in the following way. Let us assume that the beam enters the space beyond the anode through a foil. In this case the beam particles, laminar before striking the foil, collide with atoms and are scattered through a certain angle $\overline{\theta^2}$, and the directions of the velocities $\mathbf{v}_\perp$ acquired as a result of passing through the foil have a random character. If the radius of the beam is R_* close to the foil, R_0 is estimated from $\overline{\theta^2}R_*^2 \simeq iR_0^2$. The value of R_0 plays an important role in investigating the pinching of a beam.

Let us consider focusing by a longitudinal current flowing in the same direction as the beam current, where $i_0'(z)$ is an increasing function. If $i_0'(z)$ varies sufficiently slowly, Eq. (9) admits a solution with $R'' \approx 0$.

In this case we have

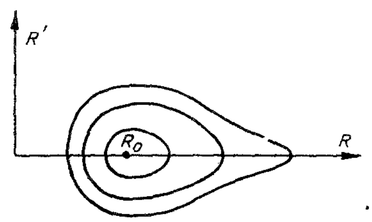

Fig. 1

$$R^2 = -\frac{ia^2}{2i_0'} + \sqrt{\left(\frac{ia^2}{2i_0'}\right)^2 + \frac{i}{i_0'} R_0^2 a^2}.$$

This relation can be confirmed by substituting into (10) if $di_0'(z)/dz \to 0$. If in addition $i_0'/i > a^2/4R_0^2$,

$$R^2 \approx aR_0 \left(\frac{i}{i_0'}\right)^{1/2}. \tag{11}$$

Estimate (11) shows that the external current has an extremely small effect on the pinching of the beam: To decrease the radius by a factor of 10 requires an external current about 10^4 times larger than the beam current. We call pinching of this type "broad." The amplitude of transverse oscillations of the beam is choked by the external field, and the frequency of oscillations increases.

After the external current is turned off the beam is characterized by the phase trajectories of Fig. 1 which show that pinching can be completely irreversible only to $R = R_0$, since in focusing to $R < R_0$ the values of R for the beam trajectory are generally larger than R_0. This fact characterizes the fundamental role of the quantity R_0 and continues to hold for any method of focusing.

We set $i_0 = -i_0' \equiv \text{const}$, and $a = R_*$, where R_* is the initial radius of the beam. Equation (10) then takes the form

$$R'' + \frac{i}{R} - \frac{i_0 R}{R_*^2} - \frac{iR_0^2}{R^3} = 0. \tag{12}$$

We call pinching described by Eq. (12) (for $i_0 \equiv \text{const} > 0$) "narrow." We note here the impossibility of "narrow" pinching by a magnetic field, since the sign of the equivalent external current can only be positive.

Since i_0 is constant, Eq. (12) has an integral of the form

$$\frac{R'^2}{2} = \frac{R_*'^2}{2} + i \ln \frac{R_*}{R} + \frac{i_0}{2}\left(\frac{R^2}{R_*^2} - 1\right) + \frac{i}{2}\left(\frac{R_0^2}{R_*^2} - \frac{R_0^2}{R^2}\right), \tag{13}$$

where R_*' is the initial value of the derivative of the radius of the beam. It follows from (13) that

$$\frac{dR'(R)}{dR} = \frac{1}{R'(R)}\left(-\frac{i}{R} + \frac{i_0 R}{R_*^2} + \frac{iR_0^2}{R^3}\right),$$

i.e., two extrema are possible for the phase curve $R'(R)$:

$$R_{1,2}^2 = \frac{i}{2i_0} R_*^2 \pm \sqrt{\left(\frac{iR_*^2}{2i_0}\right)^2 - \frac{i}{i_0} R_0^2 R_*^2}, \tag{14}$$

which is possible only for focusing by a countercurrent; for the opposite sign of the external current one of the solutions for the square of the radius would be negative.

The following types of phase trajectories are possible for Eq. (12) (Fig. 2): curve I when there are two different roots of Eq. (14) and $R'^2(R_2) > 0$; curve II when there are two roots and $R'^2(R_2) < 0$ and $R'^2(R_1) > 0$; curve III when there are no real roots of Eq. (14); curve IV describes the situation for equal roots of Eq. (14), i.e. when

$$R_*^2 = \frac{4i_0}{i} R_0^2. \tag{15}$$

Curves I and IV describe irreversible focusing when the minimum at $R = R_2$ is tangent to the straight line $R' = 0$. The best pinching, i.e., the smallest value of R_2, can be obtained for

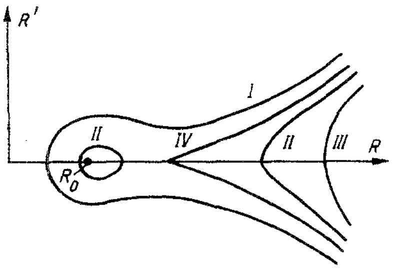

Fig. 2

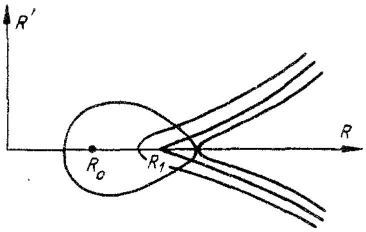

Fig. 3

equal roots $R_1 = R_2$, and therefore we consider case IV first.

Then

$$R_1^2 = R_2^2 = \frac{i}{2i_0} R_*^2 = 2R_0^2. \tag{16}$$

It follows from this that the ratio of the radii for maximum pinching is

$$\frac{R_2^2}{R_*^2} = \frac{i}{2i_0} \tag{17}$$

and the effect is possible only when $2i_0 > i$.

Substituting (16) into (13), we obtain for $R_2' = 0$

$$R_*'^2 = i \ln \frac{i}{2i_0} + i_0 - \frac{i^2}{4i_0}. \tag{18}$$

For given values of i and i_0 Eq. (18) determines the initial angle of convergence of the beam. For the model under consideration to be valid the condition $R_*' \ll 1$ must be satisfied, and this together with (18) places a restriction on the currents

$$1 \gg i \ln \frac{i}{2i_0} + i_0 - \frac{i^2}{4i_0} > 0. \tag{19}$$

The irreversible character of the focusing considered is explained by the divergence of the integral for the length Z at which the point $R' = 0$ is reached.

We have from (13)

$$Z = -\int_{R_*}^{\overline{R}} dR \Bigg/ \sqrt{i \ln \frac{R_1^2}{R^2} + \frac{i}{2}\frac{R^2}{R_1^2} - \frac{i}{2}\frac{R_1^2}{R^2}}.$$

It is clear that this integral diverges as $\overline{R} \to R_1$;

$$Z \simeq \sqrt{\frac{3}{iy}} R_1, \quad y = \frac{\overline{R} - R_1}{R_1} \ll R_1.$$

If the initial conditions for curve IV correspond to (18) (Fig. 3 shows the phase trajectories in the neighborhood of the fundamental trajectory) the focusing countercurrent can be broken off arbitrarily far away. Under real conditions, however, such a situation is unattainable; because of the unavoidable spread in R' the initial states lie in a region having a finite phase volume. Any trajectory arbitrarily close to the fundamental corresponds to a finite length Z at which the point $R' = 0$ is reached; after this the beam again begins to spread. The finite value of the integral for Z along neighboring trajectories is accounted for by the presence of the infinite derivative $dR'/dR \big|_{\substack{R \neq R_1 \\ R'=0}}$.

We show that under certain conditions irreversible pinching can occur as the result of suddenly stopping the external current.

We set $R'^2 = R_*'^2 + S(R)$ (cf. 13)) and determine $\overline{R}$ so that $R_*'^2 + S(\overline{R}) = 0$. We assume that $\overline{R}$ is only slightly different from R_1, i.e., $\overline{R} = R_1(1 + x)$ with $x \ll 1$.

We determine the distance Z_0 at which the current is stopped in the following way:

$$Z_0 = -\int_{R_*}^{R_0} \frac{dR}{\sqrt{S(R) - S(\overline{R}_0)}}, \quad \overline{R}_0 = R_1(1 + x). \tag{20}$$

Since the main contribution to the integral (20) comes from points near $\overline{R} \approx \overline{R}_0$, we expand the radicand and obtain

$$Z_0 = \frac{R_1}{|x_0|}\left(\frac{2i_0}{i^3}\right)^{1/4}. \tag{21}$$

If the initial value of R_*' differs from the value satisfying the equality $R_*'^2 + S(\overline{R}_0) = 0$, in the plane $Z = Z_0$ where the countercurrent is stopped the radius of the beam differs from $\overline{R}_0$. Assuming $\overline{R} = R_1(1 + x)$ and $\overline{\overline{R}} = R_1(1 + y)$, we have

$$Z_0 = -\int_{R_*}^{\overline{\overline{R}}} \frac{dR}{\sqrt{S(R) - S(\overline{R})}} = \frac{R_1}{x\sqrt{i}}\left(\left(\frac{2i_0}{i}\right)^{1/4} - \sqrt{y - x}\right). \tag{22}$$

A comparison of (21) and (22) gives

$$y = x + \left(\frac{x_0 - x}{x_0}\right)^2\left(\frac{2i_0}{i}\right)^{1/4}. \tag{23}$$

Equation (23) shows that there is a range of initial values for which the values of the radius in the plane where the countercurrent is stopped are negligibly different from R_1. In this case the stable region is compressed as $x_0 \to 0$, i.e., as the phase trajectory approaches the principal trajectory.

The irreversibility of pinching (the absence of an appreciable increase in the radius after the countercurrent is stopped) is accounted for by the fact that $R_1 > R_0$ and the phase trajectory of the free beam has to surround the point $R' = 0$, $R = R_0$ (Fig. 3).

It is clear from Eq. (17) that the larger the countercurrent i_0 the larger the pinch effect. The ratio of the current density of the beam after pinching j_1 and before pinching j_* is

$$j_1/j_* = 2i_0/i.$$

Since (19) shows that the maximum value of the countercurrent $i_0 \sim 1$, it follows that

$$j_1/j_* \sim 2/i, \tag{24}$$

i.e. pinching is possible only for weak-current beams ($J \ll 17\gamma\beta$ kA).

There is also another method of producing irreversible pinching when the countercurrent is suddenly stopped.

Equation (13) shows that a phase trajectory can pass through the point $R' = 0$, $R = R_0$. The initial conditions in this case must satisfy the relation

$$\frac{R_*'^2}{2} = i\ln\frac{R_0}{R_*} + \frac{i + i_0}{2}\left(1 - \frac{R_0^2}{R_*^2}\right). \tag{25}$$

The distance L from the emitter at which the radius has its minimum value $R = R_0$ is found from the integral

$$L = -\int_{R_*}^{R} \frac{dR}{\sqrt{i \ln\frac{R_0}{R} + \frac{i}{2R_*^2}(R^2 - R_0^2) + \frac{i}{2}\left(1 - \frac{R_0^2}{R_*^2}\right)}}.$$

If the countercurrent is suddenly stopped at the distance L, the radius of the beam for $z > L$ will clearly not change, since the phase trajectory of the neutralized beam degenerates into the point $R = R_0$, $R' = 0$. The stability of such a pinch is obvious, since for small deviations from this point after the countercurrent is stopped, the beam is described by a phase trajectory of correspondingly small size.

In addition to condition (25), however, the condition $R'^2(R_1) > 0$ must also be satisfied, since otherwise the phase curve is doubly-connected (type II of Fig. 2) and the points corresponding to the initial conditions and the equilibrium radius lie on different branches of the phase curve.

Using the notation $\delta = 4i_0R_0^2/iR_*^2$, $R_1^2 = (2R_0^2/\delta)\ (1 + \sqrt{1 - \delta})$ and the condition $R_1'^2 > 0$ from (13), Eq. (25) can be written in the form

$$\ln\frac{\delta}{2(1+\sqrt{1-\delta})} + \frac{\delta}{4}\left(\frac{2}{\delta}\left(1+\sqrt{1-\delta}\right) - 1\right) + 1 - \frac{\delta}{2(1+\sqrt{1-\delta})} > 0,$$

which after simple transformations reduces to

$$\ln\frac{\delta}{2(1+\sqrt{1-\delta})} - \frac{\delta}{4} + 1 + \sqrt{1-\delta} > 0.$$

This inequality is satisfied in the range $1 > \delta > 0$ if

$$\delta > \delta_* \simeq 0.86.$$

In other words, in addition to (25) the condition $4i_0R_0^2/iR_*^2 > \delta_*$ must also be satisfied. This significantly restricts the range of initial conditions for which irreversible pinching is possible.

The ratio of the current densities of the beam after and before pinching satisfies the inequality

$$j_1/j_* < \delta_* \frac{4i_0}{i}. \tag{26}$$

Equation (26) does not differ qualitatively from (24); in both cases pinching is possible for relatively weak-current beams; in the plane in which the countercurrent is stopped the maximum current density of the beam is approximately equal to the density of the countercurrent.

LITERATURE CITED

1. I. M. Kapchinskii, Dynamics of Charged Particles in Linear Resonance Accelerators [in Russian], Atomizdat, Moscow (1966).
2. J. W. Poukey and A. J. Toepfer, "Theory of superpinched relativistic electron beams," Phys. Fluids, 17, 1582 (1974).
3. E. P. Lee and R. K. Cooper, "General envelope equation for cylindrically symmetric charged-particle beams," Part. Accel., 7, 83 (1976).
4. R. K. Cooper, "Solenoid-lens effects in beam-transport equations," Part. Accel., 7, 41 (1976).
5. J. W. Poukey, A. J. Toepfer, and J. G. Kelly, "Axially dependent equilibria for a relativistic electron beam," Phys. Rev. Lett., 26, 1620 (1971).